Strong Permittivity Fluctuations[1]

The bilocal approximation used in the calculation of the ensemble average of the dyadic Green's function strictly requires that $\Delta << k_m^2$ as only the first term in the expansion of the mass operator is retained (see quation (31)). This limitation is the direct result of the singular behavior of the dyadic Green's function. To improve upon this limitation, Dyson's equation can be written for a modified integral equation. For simplicity let us assume that the medium has a spherically symmetric correlation function. Denoting the permittivity function by $\epsilon(\overline{r})$, the vector wave equation can be written as

$$\nabla \times \nabla \times \overline{E}(\overline{r}) - k_o^2 \frac{\epsilon(\overline{r})}{\epsilon_o} \overline{E}(\overline{r}) = 0 \tag{56}$$

Instead of using $\langle \epsilon(r) \rangle$ as the background medium permittivity, the effective permittivity of the medium (ϵ_e) will be used as the background permittivity. Then (56) can be written as

$$\nabla \times \nabla \times \overline{E}(\overline{r}) - k_o^2 \frac{\epsilon_e}{\epsilon_o} E(\overline{r}) = k_0^2 \left(\frac{\epsilon(\overline{r}) - \epsilon_e}{\epsilon_o} \right) \overline{E} \tag{57}$$

Suppose the dyadic Green's function for the background medium is denoted by $\overline{\overline{G}}^{e}(\overline{r}, \overline{r}')$ which satisfies

$$\nabla \times \nabla \times \overline{\overline{G}}^{e}(\overline{r}, \overline{r}') - k_e^2 \overline{\overline{G}}^{e}(\overline{r}, \overline{r}') = \overline{\overline{I}} \delta(\overline{r} - \overline{r}') \tag{58}$$

where $k_e^2 = \omega^2 \mu \epsilon_e$. Using the vector-dyadic Green's second identity in conjunction with (57) and (58), it can be shown that

$$\overline{E}(\overline{r}) = \overline{E}_e(\overline{r}) + k_o^2 \int d\overline{r}' \, \overline{\overline{G}}^{e}(\overline{r}, \overline{r}') \frac{\epsilon(\overline{r}') - \epsilon_e}{\epsilon_o} E(\overline{r}') \tag{59}$$

where $\overline{E}_e(\overline{r})$ is the mean-field in the background medium in the absence of a fluctuation given by $(\epsilon(r) - \epsilon_e)$. As shown before the dyadic Green's function for a spherically symmetric medium can be represented by

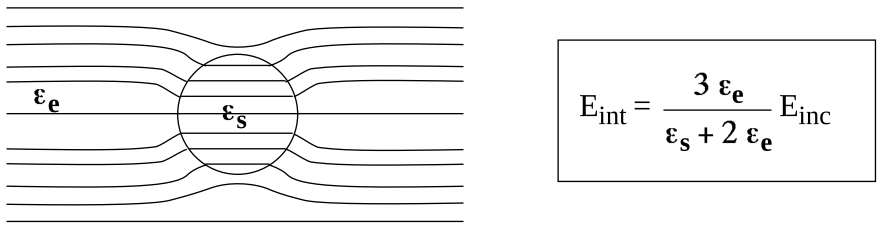

Figure 1: Potential variation near a small dielectric sphere and the relation between the incident and internal fields.

$$\overline{\overline{G}}^{e}(\overline{r},\overline{r}') = \overline{\overline{G}}^{e}_{ps}(\overline{r},\overline{r}') - \frac{\overline{\overline{I}}}{3k_e^2}\delta(\overline{r}-\overline{r}') \tag{60}$$

Substituting (60) into (59) it can be shown that

$$\overline{F}(\overline{r}) = \overline{E}_e(\overline{r}) + k_o^2 \int d\overline{r}'\, \overline{\overline{G}}^{e}_{ps}(\overline{r},\overline{r}')\xi(\overline{r})\overline{F}(\overline{r}') \tag{61}$$

where

$$\overline{F}(\overline{r}) = \frac{\epsilon(\overline{r}) + 2\epsilon_e}{3\epsilon_e}\overline{E}(\overline{r})$$

represents the excitation field for a small dielectric sphere with permittivity $\epsilon(\overline{r})$ in a background dielectric ϵ_e and internal field $\overline{E}(\overline{r})$. In (61) $\xi(\overline{r})$ is given by

$$\xi(\overline{r}) = 3\frac{\epsilon_e}{\epsilon_o}\left(\frac{\epsilon(\overline{r}) - \epsilon_e}{\epsilon(\overline{r}) + 2\epsilon_e}\right) \tag{62}$$

In the limit as $\epsilon(\overline{r}) \to \epsilon_e$ the fluctuating function $\xi(\overline{r}) \to 0$ then $\overline{F}(\overline{r}) \to \overline{E}_e(\overline{r})$. To solve (61) iteratively, $\overline{E}_e(\overline{r})$ can be substituted for $\overline{F}(\overline{r}')$ in the integral of (61), thus to the first-order

$$\overline{F}(\overline{r}) = \overline{E}_e(\overline{r}) + k_o^2 \int d\overline{r}'\, \overline{\overline{G}}^{e}_{ps}(\overline{r},\overline{r}')\xi(\overline{r}')\overline{E}_e(\overline{r}')$$

From the definition of the equivalent medium we expect that the $\langle \overline{F}(\overline{r}) \rangle$, to the first-order, reduces to $\overline{E}_e(\overline{r})$. This mandates that

$$\langle \xi(\overline{r}) \rangle = 0 \tag{63}$$

Equation (61) is analogous to (4) with $\overline{\overline{G}}_{ps}^{e}(\overline{r},\overline{r}')$ as the propagator and $k_0^2\xi(\overline{r}')$ as the scatterer. Assuming $\xi(\overline{r}')$ is Gaussian (a big assumption?!), then the bilocal approximation can be applied to (61) and we get

$$\langle\overline{F}(\overline{r})\rangle = \overline{E}_e(\overline{r}) + k_0^2 \iint d\overline{r}'d\overline{r}''\, \overline{\overline{G}}_{ps}^{e}(\overline{r},\overline{r}')\cdot \overline{\overline{\zeta}}_{eff}(\overline{r}',\overline{r}'')\langle F(\overline{r}'')\rangle \tag{64}$$

where $\overline{\overline{\zeta}}_{eff}(\overline{r}',\overline{r}'')$ is the approximate mass operator under the bilocal approximation given by

$$\overline{\overline{\zeta}}_{eff}(\overline{r}',\overline{r}'') = k_o^2\, \overline{\overline{G}}_{ps}^{e}(\overline{r}',\overline{r}'')C_\xi(|\overline{r}'-\overline{r}''|) \tag{65}$$

where

$$C_\xi(|\overline{r}'-\overline{\overline{r}}''|) = \langle\xi(\overline{r}')\xi(\overline{r}'')\rangle \tag{66}$$

In the limit as the frequency approaches zero ($k_o \to 0$) the second term of (64) vanishes. That is the scattering does not contribute to the mean field. Using (63) we have

$$\left\langle \frac{\epsilon(\overline{r})-\epsilon_e}{\epsilon(\overline{r})+2\epsilon_e}\right\rangle = 0 \tag{67}$$

Equation (67) is the fundamental equation for deriving the effective dielectric constant ϵ_e for a medium with fluctuating permittivity $\epsilon(\overline{r})$. The classical formula for the effective (equivalent) dielectric constant of mixture of particles with different permittivities, known as Polder van Santen mixing formula, can easily be obtained from (67). Suppose there are n different constituents in a mixture with permittivities and volume fractions ϵ_i and f_i respectively. Therefore

$$P[\epsilon(r)=\epsilon_i] = f_i \tag{68}$$

where $P[\cdot]$ denotes the probability noting that $\sum_{i=1}^n f_i = 1$. According to the probability distribution (68)

$$\sum_{i=1}^{n} \frac{\epsilon_i-\epsilon_e}{\epsilon_i+2\epsilon_e} f_i = 0 \tag{69}$$

For example, if there are two components in the mixture then

$$\frac{\epsilon_1-\epsilon_e}{\epsilon_1+2\epsilon_e} f_1 = \frac{\epsilon_e-\epsilon_2}{\epsilon_2+2\epsilon_e}(1-f_1) \tag{70}$$

where f_1 is the volume fraction of particle 1. Equation (70) is a second order equation for ϵ_e; however, since $\epsilon_1 \leq \epsilon_e \leq \epsilon_2$ only one of the two solutions is acceptable. Adding $-\sum_{i=1}^{n} f_i$ to the left-hand side of (69) and -1 to its right-hand side, we get

$$\sum_{i=1}^{n} \frac{f_i}{\epsilon_i + 2\epsilon_e} = \frac{1}{3\epsilon_e} \tag{71}$$

Also by adding and subtracting ϵ_o to the numerator of each term of (69) we get

$$\sum_{i=1}^{n} \frac{\epsilon_i - \epsilon_o}{\epsilon_i + 2\epsilon_e} f_i = (\epsilon_e - \epsilon_o) \sum_{i=1}^{n} \frac{f_i}{\epsilon_i + 2\epsilon_e} \tag{72}$$

Substituting (71) into (72) produces Polder van Santen mixing formula

$$\sum_{i=1}^{n} \frac{\epsilon_i - \epsilon_o}{\epsilon_i + 2\epsilon_e} f_i = \frac{\epsilon_e - \epsilon_o}{3\epsilon_e} \tag{73}$$

where ϵ_o is the free-space permittivity.

The effective permittivity given by (73) is a result of quasi-static approximation. A more general result can be obtained under the bilocal approximation. Starting from (61), the ensemble average of $\overline{F}(\overline{r})$ is given by

$$\langle \overline{F}(\overline{r}) \rangle = \overline{E}_e(\overline{r}) + k_o^2 \int d\overline{r}' \, \overline{\overline{G}}_{ps}^{e} (\overline{r}, \overline{r}') \langle \xi(\overline{r}') \overline{F}(\overline{r}') \rangle \tag{74}$$

Comparing (74) to the bilocal approximation (equation (64)) it is obvious that

$$\langle \xi(\overline{r}) \overline{F}(\overline{r}) \rangle = \int d\overline{r}' \, \overline{\overline{\xi}}_{eff} (\overline{r} - \overline{r}') \langle F(\overline{r}') \rangle \tag{75}$$

Substituting

$$\xi(\overline{r}) F(\overline{r}) = \frac{\overline{D}(\overline{r})}{\epsilon_o} - \frac{\epsilon_e}{\epsilon_o} \overline{E}(\overline{r})$$

in (75) we get

$$\langle \overline{D}(\overline{r}) \rangle = \epsilon_e \langle \overline{E}(\overline{r}) \rangle + \epsilon_o \int \overline{\overline{\xi}}_{eff} (\overline{r} - \overline{r}') \langle \overline{F}(\overline{r}') \rangle d^3 r' \tag{76}$$

The integral in (76) is of convolution type and therefore upon taking the Fourier transform of both sides we have

$$\langle \overline{D}(\overline{K}) \rangle = \epsilon_e \langle \overline{E}(\overline{K}) \rangle + \epsilon_o \, \overline{\overline{\xi}}_{eff} (\overline{K}) \langle F(\overline{K}) \rangle \tag{77}$$

Let us define the effective dielectric constant by

$$\langle \overline{D}(\overline{K})\rangle = \overline{\overline{\epsilon}}_{eff}(\overline{K})\langle \overline{E}(\overline{K})\rangle \tag{78}$$

Usually we do not expect $\overline{\overline{\epsilon}}_{eff}$ to be a function of position for a statistically homogeneous random medium, however, (77) indicates that the relation between $\langle D(\overline{K})\rangle$ and $\langle E(\overline{K})\rangle$ may be a function of $\overline{K}$. From the definition of $\overline{F}(\overline{r})$ we have

$$\langle F(\overline{r})\rangle = \frac{\langle \overline{D}(\overline{r})\rangle}{3\epsilon_e} + \frac{2}{3}\langle \overline{E}(\overline{r})\rangle \tag{79}$$

Substituting (78) and the Fourier Transform of (79) into (77), it can easily be shown that:

$$\overline{\overline{\epsilon}}_{eff}(\overline{K}) = \epsilon_e \overline{\overline{I}} + \epsilon_o \left(\overline{\overline{I}} - \frac{\epsilon_o}{3\epsilon_e}\overline{\overline{\xi}}_{eff}(\overline{K})\right)^{-1} \overline{\overline{\xi}}_{eff}(\overline{K}) \tag{80}$$

Assuming that $\xi(\overline{r})$ has small fluctuations (necessary for the validity of the bilocal approximation) then

$$\left(\overline{\overline{I}} - \frac{\epsilon_o}{3\epsilon_e}\overline{\overline{\xi}}_{eff}(\overline{K})\right)^{-1} \simeq \overline{\overline{I}}$$

and therefore

$$\overline{\overline{\epsilon}}_{eff}(\overline{K}) \simeq \epsilon_e \overline{\overline{I}} + \epsilon_o \overline{\overline{\xi}}_{eff}(\overline{K}) \tag{81}$$

Assuming $\overline{\overline{\xi}}_{eff}(\overline{K})$ is a sharply varying function (at low frequencies) we can further approximate (81) to get

$$\overline{\overline{\epsilon}}_{eff} = \epsilon_e \overline{\overline{I}} + \epsilon_o \overline{\overline{\xi}}_{eff}(0) \tag{82}$$

where

$$\begin{aligned}\overline{\overline{\xi}}_{eff}(0) &= \int d^3r\, \overline{\overline{\xi}}_{eff}(\overline{r}) \\ &= k_o^2 \int d^3r\, \overline{\overline{G}}{}^{e}_{ps}(\overline{r}) C_\xi(|\overline{r}|)\end{aligned}$$

For a spherically symmetric correlation function, using (29), and expanding $e^{ik_e r}$ in terms of its Taylor series up to second-order, we have

$$\overline{\overline{\xi}}_{eff}\ (0) \simeq \frac{2}{3} k_o^2 \left[\int_0^\infty r C_\xi(r) dr + i k_e \int_0^\infty r^2 C_\xi(r) dr \right] \tag{83}$$

Hence at low frequencies

$$\epsilon_{eff} \simeq \epsilon_e + \frac{2}{3} \epsilon_o k_o^2 A_1 + i \frac{2}{3} \epsilon_o k_o^2 k_e A_2 \tag{84}$$

where

$$\begin{aligned} A_1 &= \int_0^\infty r C_\xi(r) dr \\ A_2 &= \int_0^\infty r^2 C_\xi(r) dr \end{aligned}$$

The imaginary component of (84) represents loss due to scattering in the medium to the lowest in frequency. Note the result given by (84) is a low frequency approximation.

5 Distorted Born Approximation

The iterative solution of (61) is known as the distorted Born approximation. The only difference between the Born and distorted Born approximation is that the mean-field in the distorted Born approximation is propagating in the effective medium with some attenuation caused by scattering and the second difference is that permittivity fluctuations are considered to be around the effective dielectric constant instead of the mean dielectric constant. To demonstrate the procedure we consider a half-space random medium illuminated by a plane wave. The mean-field to the first-order in permittivity fluctuations is $\overline{E}_e(\overline{r}')$ and therefore the fluctuating component of the field outside the random medium is given by

$$f(\overline{r}) = \overline{F}(\overline{r}) - \overline{E}_e(\overline{r}) = k_o^2 \int_v d\overline{r}'\ \overline{\overline{G}}{}_{01}^{e}\ (\overline{r}, \overline{r}') \xi(\overline{r'}) \overline{E}_e(\overline{r}') \tag{85}$$

Now the scattered power can be evaluated easily

$$\begin{aligned} \langle |f(\overline{r})|^2 \rangle \;\; = \;\; & k_o^4 \iint_{-\infty}^{+\infty} dx_1 dy_1 \int_{-\infty}^{0} dz_1 \iint_{-\infty}^{+\infty} dx_2 dy_2 \int_{-\infty}^{0} dz_2\ C_\xi(|\overline{r}_1 - \overline{r}_2|) \\ & \left(\overline{\overline{G}}{}_{01}^{e}\ (\overline{r}, \overline{r}_1) \cdot E_e(\overline{r}_1 \right) \left(\overline{\overline{G}}{}_{01}^{e}\ (\overline{r}, \overline{r}_2) \cdot \overline{E}(\overline{r}_2) \right)^* \end{aligned} \tag{86}$$

Equation (86) is similar to (13) and can be solved using the procedure outlined for the first-order Born approximation.